HARMONIC CLOSE-TO-CONVEX FUNCTIONS AND MINIMAL SURFACES

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Abstract. In this paper, we study the family $\mathcal{C}_H^0$ of sense-preserving complex-valued harmonic functions f that are normalized close-to-convex functions on the open unit disk $\mathbb{D}$ with $f_{\overline{z}}(0)=0$. We derive a sufficient condition for f to belong to the class $\mathcal{C}_H^0$. We take the analytic part of f to be $zF(a,b;c;z)$ or $zF(a,b;c;z^2)$ and for a suitable choice of co-analytic part of f, the second complex dilatation $w(z)=\overline{f_{\overline{z}}}/f_z$ turns out to be a square of an analytic function. Hence f is lifted to a minimal surface expressed by an isothermal parameter. Explicit representation for classes of minimal surfaces are given. Graphs generated by using Mathematica are used for illustration.

1. Introduction and Preliminary Results

Denote by $\mathcal{H}$ the class of all complex-valued harmonic functions f in the unit disk $\mathbb{D}=\{z\in\mathbb{C}: |z|<1\}$ normalized by $f(0)=0=f_z(0)-1$, and let $\mathcal{S}_H$ be the set of univalent functions in $\mathcal{H}$. For $f\in\mathcal{H}$, we have the canonical decomposition $f=h+\overline{g}$, where g and h are analytic on $\mathbb{D}$. Here we call h the analytic part of f and g the co-analytic part of f. We have

$$h(z)=z+\sum_{n=2}^{\infty}a_nz^n \quad\text{and}\quad g(z)=\sum_{n=1}^{\infty}b_nz^n,\quad z\in\mathbb{D}$$

and the Jacobian $J_f(z)$ of f is

$$J_f(z)=|f_z(z)|^2-|f_{\overline{z}}(z)|^2=|h'(z)|^2-|g'(z)|^2.$$

We say that the function f is sense-preserving in $\mathbb{D}$ if $J_f(z)>0$ in $\mathbb{D}$. According to a result of Lewy [9], the condition $J_f(z)>0$ in $\mathbb{D}$ is necessary and sufficient for f to be locally univalent and sense-preserving. Further, $f\in\mathcal{H}$ is sense-preserving in $\mathbb{D}$ if and only if $g'(z)=\omega(z)h'(z)$, where ω is analytic in $\mathbb{D}$ with $|\omega(z)|<1$ in $\mathbb{D}$. We observe that if $g'(0)=0$, then ω fixes the origin so that by the Schwarz lemma one has $|\omega(z)|\leq|z|$ in $\mathbb{D}$.

For basic results about the theory of planar harmonic mappings we refer to [2] and the monograph of Duren [5]. A function $f\in\mathcal{H}$ is said to be convex (starlike, close-to-convex resp.) in $\mathbb{D}_r:=\{z\in\mathbb{C}: |z|<r\}$ if it univalent in $\mathbb{D}_r$ and $f(\mathbb{D}_r)$ is convex (starlike with respect to the origin, close-to-convex resp.). By $\mathcal{K}_H$, $\mathcal{S}_H^*$,

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and $\mathcal{C}_H$, we denote the subclasses of functions in $\mathcal{S}_H$ that are convex, starlike, or close-to-convex in the unit disk $\mathbb{D}$, respectively. By $\mathcal{K}_H^0$, $\mathcal{S}_H^{*0}$, and $\mathcal{C}_H^0$, we mean the respective subclasses of functions f from $\mathcal{K}_H$, $\mathcal{S}_H^*$, and $\mathcal{C}_H$ such that $f_{\overline{z}}(0) = b_1 = 0$.

One of the effective methods of constructing harmonic close-to-convex mappings from conformal mappings is based on the following result due to Clunie and Sheil-Small [2].

Lemma A. *If h, g are analytic in $\mathbb{D}$ with $|h'(0)| > |g'(0)|$ and $h + \epsilon g$ is close-to-convex for each ϵ, $|\epsilon| = 1$, then $f = h + \overline{g}$ is close-to-convex in $\mathbb{D}$.*

This lemma has been used to prove many important results. As a consequence of this result, in [13], the following result was established as a harmonic analog of Noshiro-Warschawski theorem (see [4, Theorem 2.16, p. 47]).

Lemma B. [13] *Suppose $f = h+\overline{g}$ is harmonic on $\mathbb{D}$ such that $\operatorname{Re}(e^{i\gamma}h'(z)) > |g'(z)|$ for all $z \in \mathbb{D}$, and for some $\gamma \in \mathbb{R}$. Then f is univalent, sense-preserving and close-to-convex in $\mathbb{D}$.*

In 1980, Mocanu [10] (see also [13]) proved that if $f = h+\overline{g}$ is a harmonic mapping in a convex domain Ω such that $\operatorname{Re}(h'(z)) > |g'(z)|$ for all $z \in \Omega$, then f is univalent and sense-preserving in Ω. An improved version of this results was given in [8, 13]. In order to discuss a general situation, it is appropriate to recall the following result due to Mocanu [10].

Lemma C. *Let $G \in C^1(\mathbb{D})$ be univalent such that $G(\mathbb{D})$ is a convex domain and $J_G(z) > 0$ in $\mathbb{D}$. Suppose that $F \in C^1(\mathbb{D})$, and*

$$\operatorname{Re} I(F, \overline{G}) > |I(F, G)| \quad \text{for } z \in \mathbb{D},$$

where

$$I(F, G) = \begin{vmatrix} F_z & F_{\overline{z}} \\ G_z & G_{\overline{z}} \end{vmatrix}.$$

Then F is sense-preserving and univalent in $\mathbb{D}$.

Many functions that can be proved to be univalent by using this lemma are also found to be close-to-convex in $\mathbb{D}$. In [12], the following lemma was proved which, for example, leads to the study of family of functions close-to-convex in $\mathbb{D}$ with respect to the convex function $-\log(1 - z)$.

Lemma D. *Let $f = h+\overline{g}$, where f and g are analytic in $\mathbb{D}$ such that $h(0) = g(0) = 0$ and $h'(0) = 1$. Further, let G be univalent, analytic and convex in $\mathbb{D}$. If f satisfies*

$$\operatorname{Re}\left(e^{i\theta}\frac{h'(z)}{G'(z)}\right) > \left|\frac{g'(z)}{G'(z)}\right| \quad \text{for all } z \in \mathbb{D} \text{ and for some } \theta \text{ real}, \tag{1}$$

then f is sense-preserving, harmonic, univalent and close-to-convex in $\mathbb{D}$.

The choice $G(z) = -\log(1 - z)$ leads to the family

$$\mathcal{F} = \{f \in \mathcal{H} : \operatorname{Re}\{(1 - z)f_z(z)\} > |(1 - z)f_{\overline{z}}(z)|, \ z \in \mathbb{D}\}.$$

According to Lemma 4, functions in $\mathcal{F}$ are close-to-convex in $\mathbb{D}$. As a consequence of this result, the following result was established in [12] together with some applications associated with Gaussian hypergeometric functions.

Lemma E. *Suppose that $f = h + \overline{g} \in \mathcal{H}$ satisfies the following condition*

$$\sum_{n=1}^{\infty} |(n+1)a_{n+1} - na_n| + \sum_{n=1}^{\infty} |(n+1)b_{n+1} - nb_n| \leq 1 - |b_1| \tag{2}$$

$(a_1 = 1)$. *Then $f \in \mathcal{F}$. In particular, f is harmonic close-to-convex in $\mathbb{D}$.*

In fact one can obtain the following improved version of Lemma 5.

Corollary 1. *Suppose that $f = h + \overline{g} \in \mathcal{H}$ satisfies the condition* (2). *Then $f \in \mathcal{F}_1$, where*

$$\mathcal{F}_1 = \{f \in \mathcal{H} : \, |(1-z)h'(z) - 1| < 1 - |(1-z)g'(z)|, \ z \in \mathbb{D}\},$$

and $\mathcal{F}_1 \subset \mathcal{F}$.

If we choose $G(z) = (1/2)\log((1+z)/(1-z))$ in Lemma 4, it leads to the family

$$\mathcal{F}_2 = \{f \in \mathcal{H} : \, \mathrm{Re}\,\{(1-z^2)f_z(z)\} > |(1-z^2)f_{\overline{z}}(z)|, \ z \in \mathbb{D}\}.$$

According to Lemma 4, functions in $\mathcal{F}_2$ are harmonic and close-to-convex in $\mathbb{D}$. To state a stronger version of the conclusion, it is necessary to recall the following definitions.

Definition 1. *A domain $D \subset \mathbb{C}$ is called convex in the direction α $(0 \leq \alpha < \pi)$ if every line parallel to the line through 0 and $e^{i\alpha}$ has a connected intersection with D. A univalent harmonic function f in $\mathbb{D}$ is said to be convex in the direction α if $f(\mathbb{D})$ is convex in the direction α.*

Obviously, every function that is convex in the direction α $(0 \leq \alpha < \pi)$ is necessarily close-to-convex, but the converse is not true. The class of functions convex in one direction has been studied by many mathematicians (see, for example, [3, 7, 15]) as a subclass of functions introduced by Robertson [14]. Now, we recall the following well-known result [2].

Lemma F. *A harmonic function $f = h + \overline{g}$ locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of the real axis (resp. in the direction of the imaginary axis) if and only if $h - g$ (resp. $h + g$) is a conformal univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of the real axis (resp. in the direction of the imaginary axis).*

Paul Greiner [6] has constructed numerous examples using the method of shearing. However, by using this lemma, we can easily see that functions in $\mathcal{F}_2$ are not only close-to-convex but is also convex in the vertical direction.

Lemma 1. *Functions in $\mathcal{F}_2$ are convex in the vertical direction.*

Proof. Let $F = h + g$, where $f = h + \overline{g} \in \mathcal{F}_2$. Then

$$\begin{aligned} \mathrm{Re}\,\{(1-z^2)F'(z)\} &= \mathrm{Re}\,\{(1-z^2)(h'(z)+g'(z))\} \\ &\geq \mathrm{Re}\,\{(1-z^2)h'(z)\} - |(1-z^2)g'(z)| \\ &= \mathrm{Re}\,\{(1-z^2)f_z(z)\} - |(1-z^2)f_{\overline{z}}(z)| > 0\cdot \end{aligned}$$

From Theorem 1 of [7], it is clear that F is univalent and convex in the vertical direction in $\mathbb{D}$. By [2, Theorem 5.3], it is evident that $f = h + \overline{g}$ is univalent and convex in vertical direction in $\mathbb{D}$. $\square$

Now, we present a sufficient coefficient condition for functions to be in the family $\mathcal{F}_2$.

Lemma 2. *Suppose that $f = h + \overline{g} \in \mathcal{H}$ satisfies the following condition*

$$(3) \qquad \sum_{n=1}^{\infty} |(n+1)a_{n+1} - (n-1)a_{n-1}| + \sum_{n=1}^{\infty} |(n+1)b_{n+1} - (n-1)b_{n-1}| \leq 1 - |b_1|$$

$(a_1 = 1)$. *Then $f \in \mathcal{F}_2$. In particular, f is convex in the vertical direction in $\mathbb{D}$ and hence close-to-convex in $\mathbb{D}$.*

Proof. Without loss of generality, we may assume that $g(z) \not\equiv 0$. According to Lemma 4, it suffices to show that (1) holds for some convex function G. Now, we set $G(z) = (1/2)\log((1+z)/(1-z))$. Then using (3) we find that

$$\begin{aligned} \mathrm{Re}\left(\frac{f_z(z)}{G'(z)}\right) &= \mathrm{Re}\,\{(1-z^2)h'(z)\} \\ &= \mathrm{Re}\left(1 + \sum_{n=1}^{\infty} \big((n+1)a_{n+1} - (n-1)a_{n-1}\big) z^n \right) \\ &\geq 1 - \sum_{n=1}^{\infty} |(n+1)a_{n+1} - (n-1)a_{n-1}| \\ &\geq |b_1| + \sum_{n=1}^{\infty} |(n+1)b_{n+1} - (n-1)b_{n-1}| \\ &> \left| b_1 + \sum_{n=1}^{\infty} \big((n+1)b_{n+1} - (n-1)b_{n-1}\big) z^n \right| \\ &= |(1-z^2)g'(z)| = \left|\frac{f_{\overline{z}}(z)}{G'(z)}\right|. \end{aligned}$$

The desired conclusion follows from Lemma 4. $\square$

It is well known that the Euclidean coordinates of a minimal surface are harmonic functions of isothermal parameters. The projection of such surface onto the base plane defines a harmonic mapping. Conversely, a harmonic mapping which can be lifted to a minimal surface has a simple representation. Weierstrass-Enneper representation (see [5, p.177, Theorem]) given below describes the relation between

a minimal surface defined by isothermal parameters and the corresponding harmonic mappings.

Theorem G. *If a minimal graph* $\{(u,v,F(u,v)) : u+iv \in \Omega\}$ *is parameterized by sense-preserving isothermal parameters* $z = x+iy \in \mathbb{D}$*, the projection onto its base plane defines a harmonic mapping* $w = u+iv = f(z)$ *of* $\mathbb{D}$ *onto* Ω *whose dilatation is the square of an analytic function. Conversely, if* $f = h+\overline{g}$ *is a sense-preserving harmonic mapping of* $\mathbb{D}$ *onto some domain* Ω *with dilatation* $w = q^2$ *for some function* q *analytic in* $\mathbb{D}$*, then the formulas*

$$u = \mathrm{Re}\,(f(z)),\ v = \mathrm{Im}\,(f(z)),\ t = 2\mathrm{Im}\left\{\int_0^z q(\zeta)h'(\zeta)\,\mathrm{d}\zeta\right\} \tag{4}$$

define by isothermal parameters a minimal graph whose projection is f*. Except for the choice of sign and an arbitrary additive constant in the third coordinate function, this is the only such surface.*

As an application, we consider a particular type of analytic part of f, involving the Gaussian hypergeometric function $F(a,b;c;z)$, and a suitable co-analytic part of f so that dilatation $\omega(z)$ turns out to be a constant multiple of z^n, $n \in \mathbb{N}$. When n is an even number, the dilatation is a square of an analytic function, hence the harmonic mapping can be lifted to a minimal surface expressed by isothermal parameters. Recently in [11], the authors considered certain class of harmonic mappings convex in the horizontal direction with suitable dilatations ω and discussed the minimal surfaces associated with these harmonic mappings.

2. Applications

For complex numbers a, b and c with $c \neq 0, -1, -2, \ldots$, the *Gaussian* hypergeometric function defined by the series

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)(1,n)} z^n$$

is analytic in $|z| < 1$, where $(a,0) = 1$ for $a \neq 0$, and $(a,n) = a(a+1)(a+2)\cdots(a+n-1)$ for $n \in \mathbb{N} = \{1,2,\ldots\}$. For $\mathrm{Re}\,a > 0$ and $\mathrm{Re}\,b > 0$, we use the beta function $B(a,b)$ defined by

$$B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)},$$

where $\Gamma(a)$ is the usual gamma function. In what follows, we need the Stirling formula [1, p.57, Equation (5)] given by

$$\lim_{n\to\infty} \frac{(a,n)(b,n)}{(c,n)(1,n)} = \begin{cases} \dfrac{\Gamma(c)}{\Gamma(a)\Gamma(b)} & \text{if } c+1 = a+b, \\ 0 & \text{if } c+1 > a+b, \\ \infty & \text{if } c+1 < a+b. \end{cases} \tag{5}$$

3. Main Results

Theorem 1. *Let $a>0$, $b>0$, or $a\in\mathbb{C}\setminus\{0\}$ with $b=\overline{a}$, $\mathrm{Re}\,a>0$. Suppose that a, b, $m\in\mathbb{N}$ and $\alpha\in\mathbb{C}$ are related by any one of the following conditions:*

$$ab\leq 1 \quad \text{and} \quad |\alpha|(2B(a,b)-1)\leq 1, \tag{6}$$

$$ab\geq \max\left\{1,\frac{a+b}{2}\right\} \quad \text{and} \quad |\alpha|\leq 2B(a,b)-1. \tag{7}$$

Then the harmonic function f given by

$$f(z)=zF(a,b;a+b;z)+\overline{\frac{\alpha z^{m+1}}{m+1}\left\{F(a,b;a+b;z)*F(2,m+1;m+2;z)\right\}} \tag{8}$$

belongs to the class $\mathcal{F}_1$ (and hence, is close-to-convex in $\mathbb{D}$ with respect to $-\log(1-z)$). Moreover, the dilatation of $f(z)$ is αz^m.

Proof. Following the standard notation, we let $f(z)=h(z)+\overline{g(z)}$, where

$$h(z)=zF(a,b;a+b;z)=\sum_{n=1}^{\infty}A_nz^n$$

and

$$g(z)=\frac{\alpha z^{m+1}}{m+1}\left\{F(a,b;a+b;z)*F(2,m+1;m+2;z)\right\}=\frac{\alpha z^m}{m+1}\sum_{n=1}^{\infty}C_nz^n,$$

with

$$A_n=\frac{(a,n-1)(b,n-1)}{(a+b,n-1)(1,n-1)},\quad n\geq 1,$$

and

$$C_n=A_n\frac{(2,n-1)(m+1,n-1)}{(m+2,n-1)(1,n-1)},\quad n\geq 1.$$

Here $*$ denotes the usual Hadamard product (convolution) of power series. We see that $A_1=1=C_1$, $A_n>0$ and $C_n>0$ for all $n\geq 1$. Now, $f(z)$ takes the form

$$f(z)=\sum_{n=1}^{\infty}A_nz^n+\frac{\overline{\alpha}}{m+1}\sum_{n=m+1}^{\infty}B_n\overline{z^n},$$

so that $B_1=B_2=\cdots=B_m=0$ and $B_n=C_{n-m}$ for $n\geq m+1$. Further, a simple calculation gives

$$nA_n-(n+1)A_{n+1}=\frac{A_n}{n(a+b+n-1)}X(n),\quad n\geq 1,$$

where

$$X(n)=(n-1)(1-ab)+a+b-2ab.$$

Similarly, a computation shows that for $n\geq m+1$,

$$nB_n-(n+1)B_{n+1}=\frac{nB_n}{(n-m)^2(a+b+n-m-1)}Y(n),$$

where

$$Y(n)=(n-m-1)(1-ab)+a+b-2ab.$$

In order to apply Corollary 1 together with (2), it is convenient to write

$$T := T_1 + (|\alpha|/(m+1))T_2,$$

where

$$T_1 = \sum_{n=1}^{\infty} |(n+1)A_{n+1} - nA_n| \ \text{ and } \ T_2 = \sum_{n=1}^{\infty} |(n+1)B_{n+1} - nB_n|,$$

with $B_1 = B_2 = \cdots = B_m = 0$. Clearly, by Corollary 1, $f \in \mathcal{F}_1$ (and hence, f is close-to-convex with respect to $-\log(1-z)$) if $T \le 1$. Thus, to complete the proof, it suffices to show that $T \le 1$ under the hypotheses of the theorem.

Case **(a)**: Suppose that $1 - ab \ge 0$. Then

$$\frac{1}{a} + \frac{1}{b} \ge \frac{1}{a} + a \ge 2,$$

and so $a + b - 2ab \ge 0$. In view of this observation and (6), it is clear that $X(n) \ge X(1) \ge 0$ for all $n \ge 1$. Similarly, $Y(n) \ge Y(m+1) \ge 0$ for $n \ge m+1$. Thus, T_1 can be written as

$$\begin{aligned} T_1 &= \lim_{k\to\infty} \sum_{n=1}^{k} \big(nA_n - (n+1)A_{n+1}\big) \\ &= 1 - \lim_{k\to\infty} (k+1)A_{k+1} \\ &= 1 - \lim_{k\to\infty} \left(k\frac{(a,k)(b,k)}{(a+b,k)(1,k)} + \frac{(a,k)(b,k)}{(a+b,k)(1,k)} \right) \\ &= 1 - \frac{ab}{a+b} \lim_{k\to\infty} \frac{(a+1,k-1)(b+1,k-1)}{(a+b+1,k-1)(1,k-1)} - \lim_{k\to\infty} \frac{(a,k)(b,k)}{(a+b,k)(1,k)}. \end{aligned}$$

Thus, by the Stirling formula (5), we get

$$T_1 = 1 - \frac{ab}{a+b}\frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)} = 1 - \frac{1}{B(a,b)}. \tag{9}$$

Next, as $B_1 = B_2 = \cdots = B_m = 0$ and $Y(n) \ge 0$ for all $n \ge 2$, we have

$$\begin{aligned} T_2 &= |(m+1)B_{m+1} - mB_m| + \sum_{n=m+1}^{\infty} \big|(n+1)B_{n+1} - nB_n\big| \\ &= (m+1) + \lim_{k\to\infty} \sum_{n=m+1}^{k} \big(nB_n - (n+1)B_{n+1}\big) \\ &= (m+1) + \lim_{k\to\infty} \big((m+1) - (k+1)B_{k+1}\big) \\ &= 2(m+1) - \lim_{k\to\infty} (k+1)B_{k+1}. \end{aligned}$$

In order to compute the limit on the right hand side, we rewrite $(k+1)B_{k+1}$ as

$$\begin{aligned}(k+1)B_{k+1} &= (k-m)\frac{(a,k-m)(b,k-m)}{(a+b,k-m)(1,k-m)}\frac{(2,k-m)(m+1,k-m)}{(m+2,k-m)(1,k-m)}\\ &\quad +(m+1)\frac{(a,k-m)(b,k-m)}{(a+b,k-m)(1,k-m)}\frac{(2,k-m)(m+1,k-m)}{(m+2,k-m)(1,k-m)}.\end{aligned}$$

Applying the Stirling formula (5) as above, we easily obtain that

$$T_2 = 2(m+1) - \frac{m+1}{B(a,b)}. \tag{10}$$

Combining (9) and (10), we have

$$T = 1 + 2|\alpha| - \frac{1+|\alpha|}{B(a,b)}.$$

Under the hypothesis (6), it is clear that $T \leq 1$ and therefore $f \in \mathcal{F}_1$.

Case **(b)**: Suppose that (7) holds. Then $ab \geq 1$ and $2ab \geq a+b$ so that $X(n) \leq X(1) \leq 0$ for all $n \geq 1$. Similarly, it follows that for all $n \geq m+1$, $Y(n) \leq Y(m+1) \leq 0$. Consequently, as in the proof of Case **(a)**, the sum T takes the form

$$\begin{aligned} T &= \lim_{k\to\infty}\sum_{n=1}^{k}\big((n+1)A_{n+1}-nA_n\big) + \frac{|\alpha|}{m+1}\lim_{k\to\infty}\sum_{n=m}^{k}\big((n+1)B_{n+1}-nB_n\big)\\ &= \left(\frac{1}{B(a,b)}-1\right) + \frac{|\alpha|}{B(a,b)}.\end{aligned}$$

Therefore $T \leq 1$ holds and thus, it follows that $f \in \mathcal{F}_1$ under the condition (7) and the above relation.

Finally, from the power series representation of $h(z)$ and $g(z)$ it is easy to see that

$$g'(z) = \frac{\alpha}{m+1}\sum_{n=1}^{\infty}\frac{(a,n-1)(b,n-1)}{(a+b,n-1)(1,n-1)}\frac{(2,n-1)(m+1,n-1)}{(m+2,n-1)(1,n-1)}(n+m)z^{n+m-1}.$$

Since $(a,n)(a+n) = a(a+1,n)$, we may rewrite the last series as

$$g'(z) = \alpha z^m \sum_{n=1}^{\infty}\frac{(a,n-1)(b,n-1)}{(a+b,n-1)(1,n-1)}nz^{n-1} = \alpha z^m h'(z).$$

Therefore the dilatation of $f(z)$ is $\omega(z) = g'(z)/h'(z) = \alpha z^m$. ☐

Corollary 2. *Assume the hypotheses of Theorem 1 on a, b, m and α. In addition, if $m = 2k$, then the formula $(\mathrm{Re}\,(f(z)), \mathrm{Im}\,(f(z)), t(z))$ defines a minimal surface, where*

$$f(z) = zF(a,b;a+b;z) + \overline{\frac{\alpha z^{2k+1}}{2k+1}\Big\{F(a,b;a+b;z) * F(2,2k+1;2k+2;z)\Big\}}$$

and

$$t(z) = 2\mathrm{Im}\left\{\frac{\sqrt{\alpha}z^{k+1}}{k+1}[F(a,b;a+b;z) * F(2,k+1;k+2;z)]\right\} + c,\ \ c \in \mathbb{R},$$

whose projection is $f(z)$.

Proof. Theorem 1 gives that $f(z)$ is univalent in $\mathbb{D}$. From the definition of $f(z)$, we see that the second complex dilatation $\omega(z) = q^2(z) = \alpha z^{2k}$, which is a square of an analytic function. Therefore by the Weierstrass-Enneper theorem (see Theorem 7), the function f can be lifted to a minimal surface using the formula (4). By considering the power series representation of $h(z)$ it is easy to see that $t(z)$ takes the form

$$t(z) = 2\mathrm{Im}\left\{\frac{\sqrt{\alpha}z^{k+1}}{k+1}[F(a,b;a+b;z) * F(2,k+1;k+2;z)]\right\} + c,\ \ c\in\mathbb{R}.$$

This completes the proof. □

In the case $a, b > 0$, we may reformulate Theorem 1 in the following form.

Corollary 3. *Let $a > 0, b > 0, m \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ satisfies any one of the following conditions*

$$a\in(0,\infty),\ \ b\in\left(0,\frac{1}{a}\right]\ \ \text{and}\ \ |\alpha|(2B(a,b)-1)\le 1,$$

$$a\in\left(\frac{1}{2},\infty\right),\ \ b\in\left[\frac{a}{2a-1},\infty\right)\ \ \text{and}\ \ |\alpha|\le 2B(a,b)-1.$$

Then the harmonic function $f(z)$ defined in (8) is close-to-convex in $\mathbb{D}$.

Example 1. *If we let $a = 1$, and $b = 1$ in Corollaries 2 and 3, then we have the following: If $m \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ such that $0 < |\alpha| \le 1$, then the function*

$$f(z) = zF(1,1;2;z) + \overline{\frac{\alpha z^m}{m+1}\int_0^z F(2,m+1;m+2;t)\,dt}$$

belongs to $\mathcal{F}_1$. Using the derivative formula

$$F(a+1,b+1;c+1;z) = \frac{c}{ab}F'(a,b;c;z),$$

the above integral can be computed and as a consequence, we conclude that

$$f(z) = -\log(1-z) + \overline{\frac{\alpha z^m}{m}(F(1,m;m+1;z)-1)}$$

belongs to $\mathcal{F}_1$. In particular,

(11)
$$f(z) = \begin{cases} -\log(1-z) - \overline{\alpha(z+\log(1-z))} & \text{if } m=1,\\ -\log(1-z) - \overline{(\alpha/2)(2z+z^2+2\log(1-z))} & \text{if } m=2,\\ -\log(1-z) - \overline{(\alpha/6)(6z+3z^2+2z^3+6\log(1-z))} & \text{if } m=3,\\ -\log(1-z) - \overline{(\alpha/12)(12z+6z^2+4z^3+3z^4+12\log(1-z))} & \text{if } m=4,\end{cases}$$

and

$$f(z) = -\log(1-z) - \overline{(\alpha/60)(60z+30z^2+20z^3+15z^4+12z^5+10z^6+60\log(1-z))}$$

for $m=6$. Especially, when $m=2,4,6$ the above harmonic mappings can be lifted to minimal surface in $\mathbb{R}^3$ whose coordinates are given by the formula $(\mathrm{Re}\,(f(z)), \mathrm{Im}\,(f(z)), t(z))$ *where*

$$t(z)=\begin{cases} -2\mathrm{Im}\,\{\sqrt{\alpha}(z+\log(1-z))\} & \text{if } m=2,\\ -\mathrm{Im}\,\{\sqrt{\alpha}(2z+z^2+2\log(1-z))\} & \text{if } m=4,\\ -(1/3)\mathrm{Im}\,\{\sqrt{\alpha}(6z+3z^2+2z^3+6\log(1-z))\} & \text{if } m=6.\end{cases}$$

The images of the disk $|z|<r$ for r closer to 1 under $f(z)$ in (11) for certain values of m, α and the corresponding minimal surfaces are shown in Figures 1**(a)**-**(c)**. These figures are drawn by using Mathematica.

Theorem 2. *Let $a>0$, $b>0$, or $a\in\mathbb{C}\setminus\{0\}$ with $b=\overline{a}$, $\mathrm{Re}\,a>0$. Suppose that a, b, $m\in\mathbb{N}$ and $\alpha\in\mathbb{C}$ are related by any one of the following conditions:*

$$ab\leq \min\left\{\frac{1}{2},\frac{a+b}{3}\right\} \quad \text{and} \quad |\alpha|(B(a,b)-1)\leq 1, \tag{12}$$

$$ab\geq \max\left\{\frac{1}{2},\frac{a+b}{3}\right\} \quad \text{and} \quad |\alpha|\leq B(a,b)-1. \tag{13}$$

Then the harmonic function f given by

$$f(z)=zF(a,b;a+b;z^2)+\overline{\frac{\alpha z^{m+1}}{m+1}\left\{F(a,b;a+b;z^2)*F(2,m+1;m+2;z)\right\}} \tag{14}$$

belongs to $\mathcal{F}_2$ (and hence, f is convex in the vertical direction). Moreover, the dilatation of $f(z)$ is αz^m.

Proof. As in the proof of Theorem 1, we let $f(z)=h(z)+\overline{g(z)}$, where

$$h(z)=zF(a,b;a+b;z^2)=\sum_{n=1}^{\infty}A_{2n-1}z^{2n-1}$$

and

$$g(z)=\frac{\alpha z^{m+1}}{m+1}\left\{F(a,b;a+b;z^2)*F(2,m+1;m+2;z)\right\}=\frac{\alpha z^m}{m+1}\sum_{n=1}^{\infty}C_{2n-1}z^{2n-1}$$

with

$$A_{2n-1}=\frac{(a,n-1)(b,n-1)}{(a+b,n-1)(1,n-1)},\quad n\geq 1,$$

and

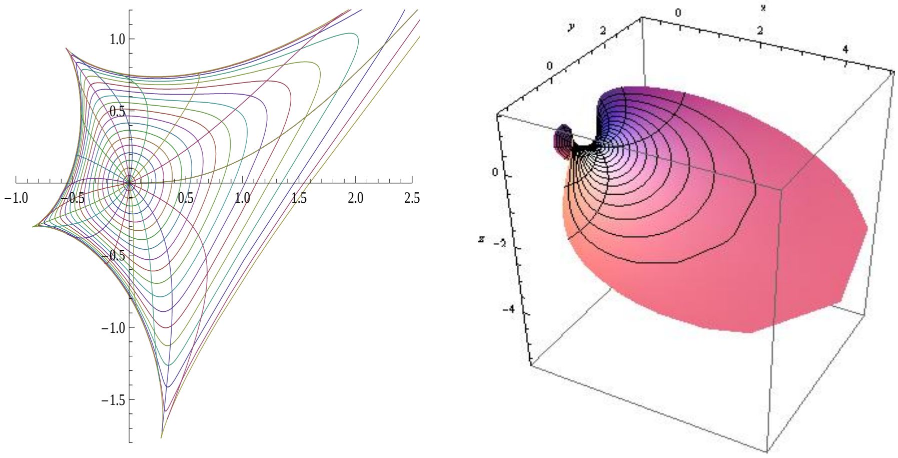
$$C_{2n-1}=A_{2n-1}\frac{(2,2n-2)(m+1,2n-2)}{(m+2,2n-2)(1,2n-2)},\quad n\geq 1.$$

For convenience, we set $C_{2n-1}=B_{2n+m-1}$ so that

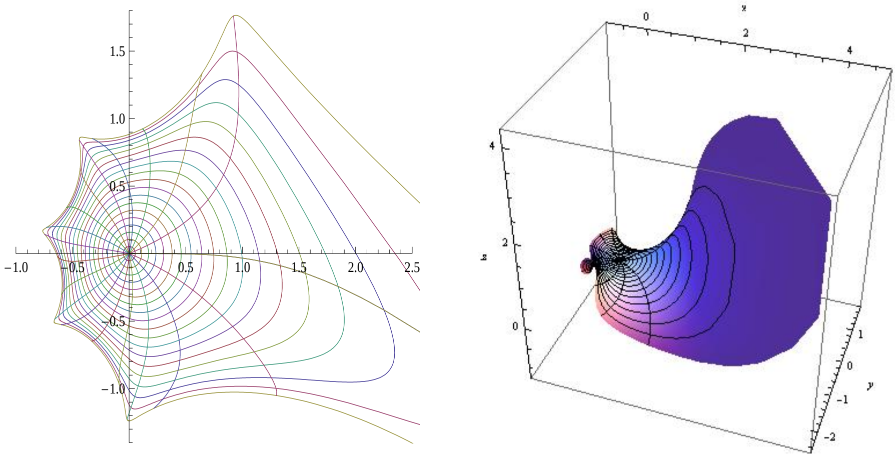
$$f(z)=\sum_{n=1}^{\infty}A_{2n-1}z^{2n-1}+\overline{\frac{\alpha}{m+1}\sum_{n=1}^{\infty}B_{2n+m-1}z^{2n+m-1}}.$$

A simple calculation gives

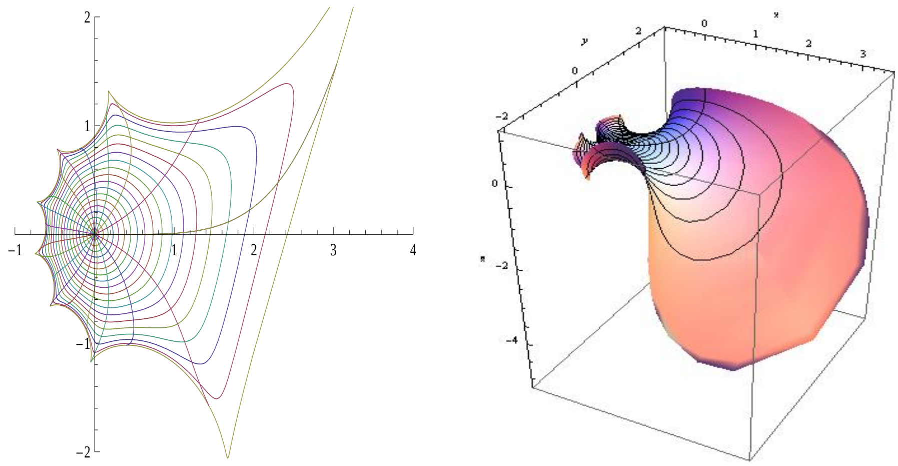
$$(2n-1)A_{2n-1}-(2n+1)A_{2n+1}=\frac{A_{2n-1}}{n(a+b+n-1)}X(n),\quad n\geq 1$$



(a) $m = 2$ and $\alpha = -i$



(b) $m = 4$ and $\alpha = 0.75i$



(c) $m = 6$ and $\alpha = -\frac{(1-i)}{\sqrt{2}}$

FIGURE 1. Images of $f(z)$ and the corresponding minimal surfaces for the indicated values of m and α

where

$$X(n) = (n-1)(1-2ab) + a + b - 3ab.$$

and by assumption (12), $X(n) \geq 0$ is satisfied. Similarly, for $n \geq 1$,

$$(2n+m-1)B_{2n+m-1} - (2n+m+1)B_{2n+m+1} = \frac{(2n+m-1)B_{2n+m-1}}{n(2n-1)(a+b+n-1)}X(n).$$

Again, let $T := T_1 + (|\alpha|/(m+1))T_2$, where

$$T_1 = \sum_{n=1}^{\infty} |(n+1)A_{n+1} - (n-1)A_{n-1}| \text{ and } T_2 = \sum_{n=1}^{\infty} |(n+1)B_{n+1} - (n-1)B_{n-1}|,$$

with $B_1 = B_2 = \cdots = B_m = 0$. By Lemma 2, it suffices to show that $T \leq 1$.

Following the proof of Theorem 1, the series T_1 and T_2 may be computed easily and see that

$$T = 1 - \frac{2}{B(a,b)} + 2|\alpha| - \frac{2|\alpha|}{B(a,b)}.$$

Under the hypothesis (12), $T \leq 1$ and therefore $f \in \mathcal{F}_2$.

Similarly, when (13) holds, $X(n) \leq 0$ so that T reduces to

$$T = \frac{2}{B(a,b)} - 1 + \frac{2|\alpha|}{B(a,b)}.$$

Therefore the conclusion $T \leq 1$ follows from (13) and the above relation.

Finally, from the power series representation of $h(z)$ and $g(z)$ it is easy to see that

$$g'(z) = \frac{\alpha}{m+1}\sum_{n=1}^{\infty} \frac{(a,n-1)(b,n-1)}{(a+b,n-1)(1,n-1)} \frac{(2,2n-2)(m+1,2n-2)}{(m+2,2n-2)(1,2n-2)}(2n+m-1)z^{2n+m-2}.$$

Since $(a,n)(a+n) = a(a+1,n)$, we may rewrite the last series as

$$g'(z) = \alpha z^m \sum_{n=1}^{\infty} \frac{(a,n-1)(b,n-1)}{(a+b,n-1)(1,n-1)}(2n-1)z^{2n-2} = \alpha z^m h'(z).$$

Therefore the dilatation of $f(z)$ is $\omega(z) = g'(z)/h'(z) = \alpha z^m$. □

Corollary 4. *Suppose a, b, m and α satisfies the hypothesis of Theorem 1 and further $m = 2k$. Then the formula $(\mathrm{Re}\,(f(z)), \mathrm{Im}\,(f(z)), t(z))$ defines a minimal surface, where*

$$f(z) = zF(a,b;a+b;z^2) + \overline{\frac{\alpha z^{2k+1}}{2k+1}\left\{F(a,b;a+b;z^2) * F(2,2k+1;2k+2;z)\right\}}$$

and

$$t(z) = 2\mathrm{Im}\left\{\frac{\sqrt{\alpha}z^{k+1}}{k+1}[F(a,b;a+b;z^2) * F(2,k+1;k+2;z)]\right\} + c,\ \ c \in \mathbb{R},$$

whose projection is $f(z)$.

Proof. The result follows from the previous theorem and the Weierstrass-Enneper representation for minimal surface whose coordinates (u, v, t) in $\mathbb{R}^3$ is given by

$$u = \mathrm{Re}\,(f(z)),\ v = \mathrm{Im}\,(f(z)),\ t(z) = 2\mathrm{Im}\left\{\int_0^z q(\zeta)h'(\zeta)\,\mathrm{d}\zeta\right\}.$$

By considering the power series representation of $h(z)$ it is easy to see that $t(z)$ takes the form

$$t(z) = 2\mathrm{Im}\left\{\frac{\sqrt{\alpha}z^{k+1}}{k+1}[F(a, b; a+b; z^2) * F(2, k+1; k+2; z)]\right\} + c,\ c \in \mathbb{R}.$$

This completes the proof. □

Remark 1. From Lemma 1 it is clear that the function $f(z)$ in (14) is not only close-to-convex in $\mathbb{D}$, but also convex in the vertical direction in $\mathbb{D}$. Using [2, Theorem 5.3], it is easy to see that the conformal pre-shear of $f(z)$ defined by

$$\phi(z) = h(z)+g(z) = zF(a, b; a+b; z^2)+\frac{\alpha z^{m+1}}{m+1}\left\{F(a, b; a+b; z^2)*F(2, m+1; m+2; z)\right\}$$

is univalent and convex in the vertical direction in $\mathbb{D}$.

Corollary 5. *Let $a > 0$, $b > 0$ be such that*

$$b \in \begin{cases} \left(0, \dfrac{1}{2a}\right] & \text{if } a \in (0, \frac{1}{2}] \cup [1, \infty), \\ \left(0, \dfrac{a}{3a-1}\right] & \text{if } a \in [\frac{1}{2}, 1], \end{cases}$$

and $\alpha \in \mathbb{C}$ satisfies the condition $|\alpha|(B(a, b) - 1) \leq 1$. Then the function $f(z)$ defined in (14) belongs to the class $\mathcal{F}_2$.

Proof. In order to prove the result, it suffices to show that the above conditions imply (12). When $0 < b \leq (1/2a)$, it is clear that $ab \leq 1/2$. If $0 < a < 1/3$, then $3a - 1 < 0$ and hence $3ab < a + b$. We have

$$b \leq \frac{1}{2a} \leq \frac{a}{3a-1},\quad \text{if}\ \ a \in \left(\frac{1}{3}, \frac{1}{2}\right] \cup [1, \infty),$$

which again implies $3ab \leq a + b$. When a=1/3, the inequality $3ab \leq a + b$ holds.

$$b \leq \frac{a}{3a-1} \leq \frac{1}{2a}\ \ \text{if}\ \ a \in \left[\frac{1}{2}, 1\right].$$

Combining these observations, we get the result. □

Corollary 6. *Let $a > 0$, $b > 0$ be such that*

$$b \in \begin{cases} \left[\dfrac{a}{3a-1}, \infty\right) & \text{if } a \in (\frac{1}{3}, \frac{1}{2}] \cup [1, \infty), \\ \left[\dfrac{1}{2a}, \infty\right) & \text{if } a \in [\frac{1}{2}, 1], \end{cases}$$

and $\alpha \in \mathbb{C}$ satisfies the condition $|\alpha| \le B(a,b) - 1$. Then the function $f(z)$ defined in (14) *belongs to the class $\mathcal{F}_2$.*

Proof. The condition

$$ab \ge \max\left\{\frac{1}{2}, \frac{a+b}{3}\right\}$$

can be rewritten as

$$b \ge \max\left\{\frac{1}{2a}, \frac{a}{3a-1}\right\} = \begin{cases} \dfrac{a}{3a-1} & \text{for } (a-1)(a-1/2) \ge 0, \\ \dfrac{1}{2a} & \text{for } (a-1)(a-1/2) \le 0. \end{cases}$$

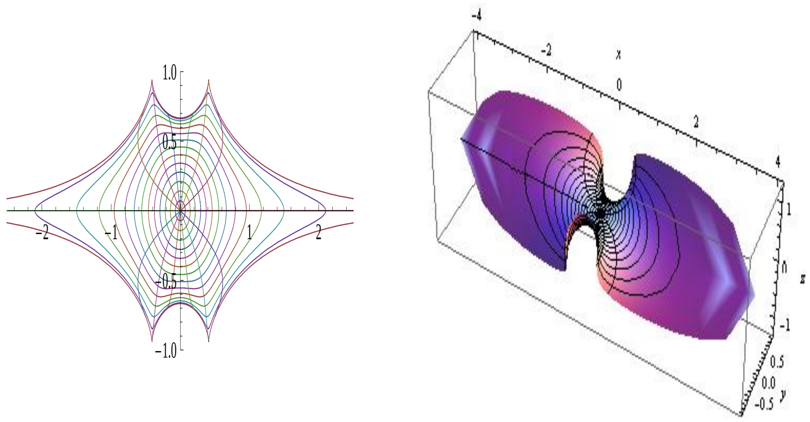
Since a is positive, $a/(3a-1) > 0$ if $a > 1/3$. The result follows from the above inequality and (13). □

Example 2. *If $b \in (0, 1/2)$, $m \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ such that $0 < |\alpha| \le b/(1-b)$, then the function*

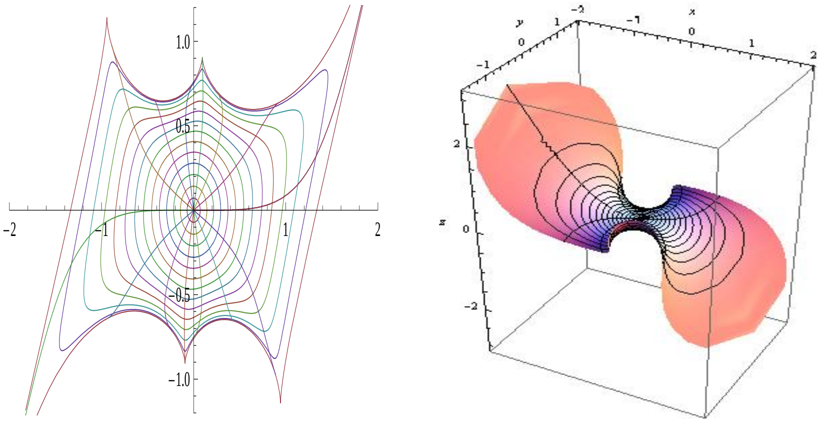
$$f(z) = zF(1,b;1+b;z^2) + \overline{\frac{\alpha z^{m+1}}{m+1}\left\{F(1,b;1+b;z^2) * F(2,m+1;m+2;z)\right\}}$$

belongs to the class $\mathcal{F}_2$ and hence f is close-to-convex in $\mathbb{D}$.

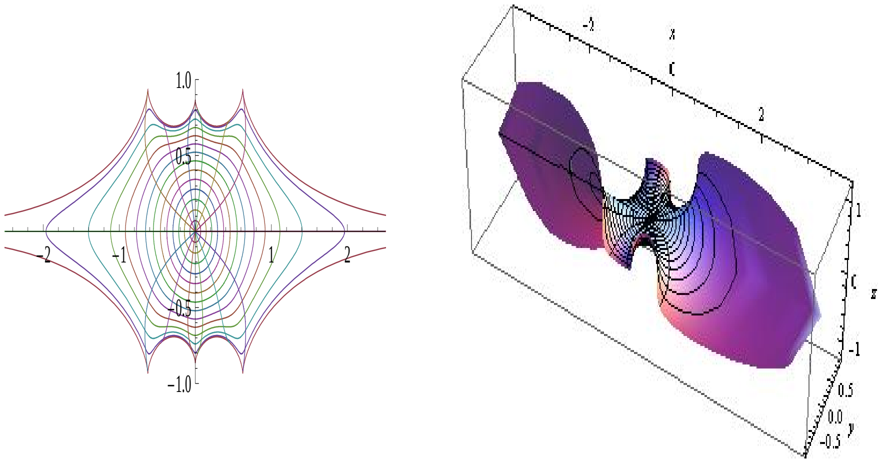
The images of the disk $|z| < r$ for r closer to 1 under $f(z)$ in (14) for certain values of a, b, m, α and the corresponding minimal surfaces are shown in Figures 2**(a)**-**(e)**. As mentioned in the Remark 1 functions present in the class $\mathcal{F}_2$ are convex in the vertical direction.



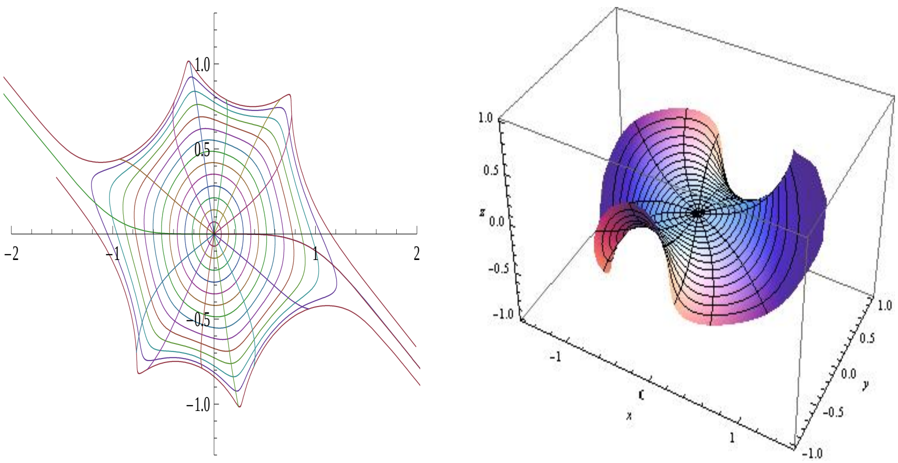
(a) $a = 1, b = 0.5, m = 4$ and $\alpha = 1$



(b) $a = 1, b = 0.5, m = 4$ and $\alpha = -\frac{(1+i)}{\sqrt{2}}$

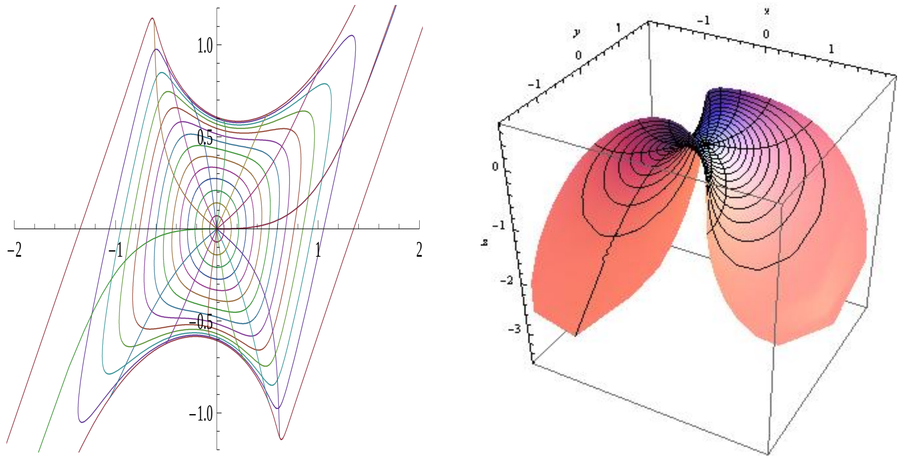


(c) $a = 1, b = 0.5, m = 6$ and $\alpha = 1$



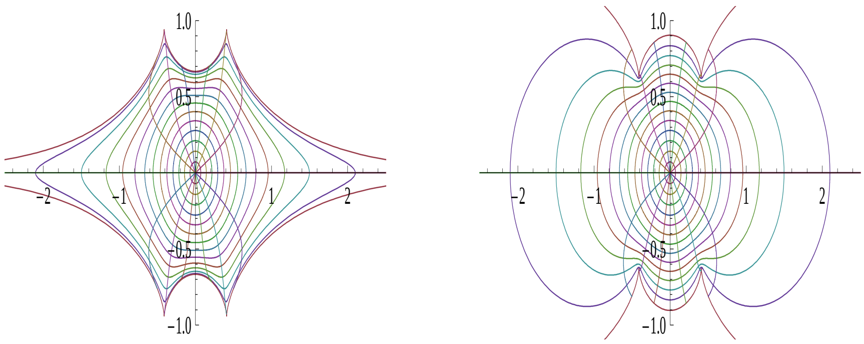
(d) $a = 1, b = \frac{1}{7}, m = 4$ and $\alpha = \frac{i}{7}$

The image of the harmonic mappings in Figures 2**(a)**-**(e)** together with the corresponding conformal pre-shears are shown in Figures 3**(a)**-**(e)**.

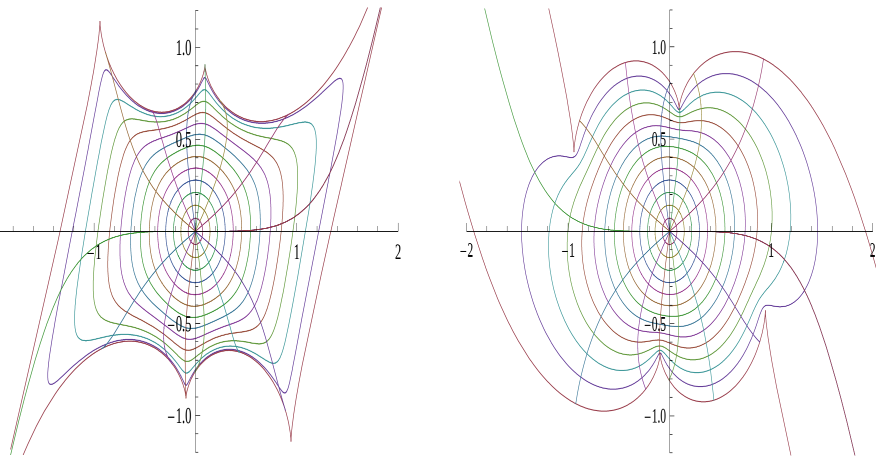


(e) $a = \frac{3}{4}, b = \frac{2}{3}, m = 2$ and $\alpha = -\frac{0.871(1+i)}{\sqrt{2}}$

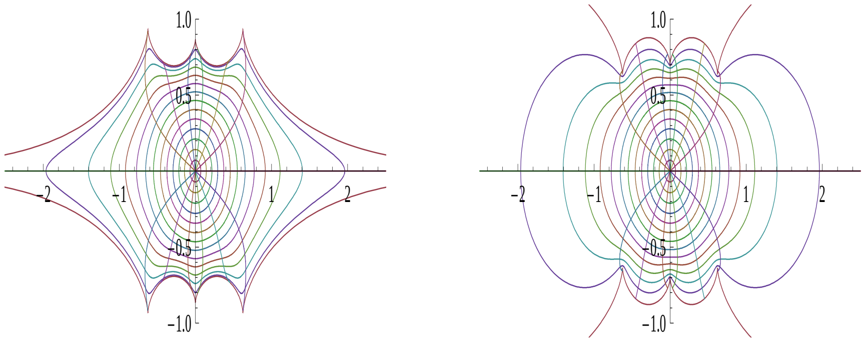
FIGURE 2. Images of $f(z)$ and the corresponding minimal surfaces for the indicated values of a, b, m and α



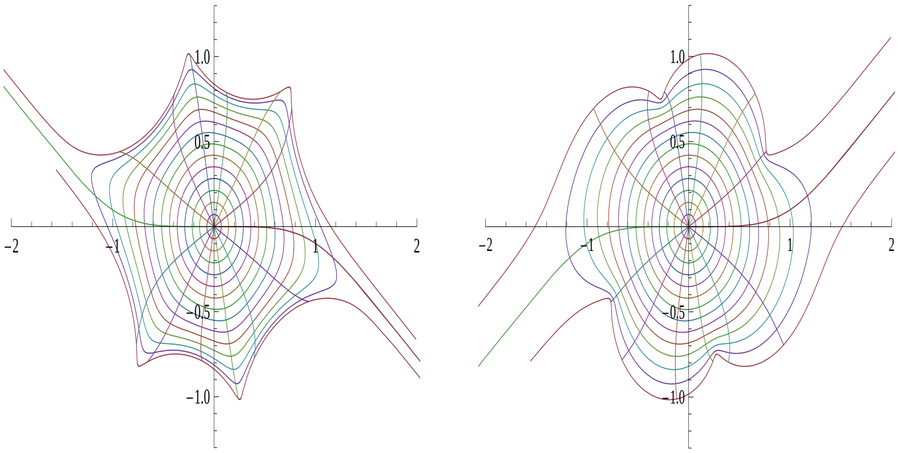
(a) $a = 1, b = 0.5, m = 4$ and $\alpha = 1$



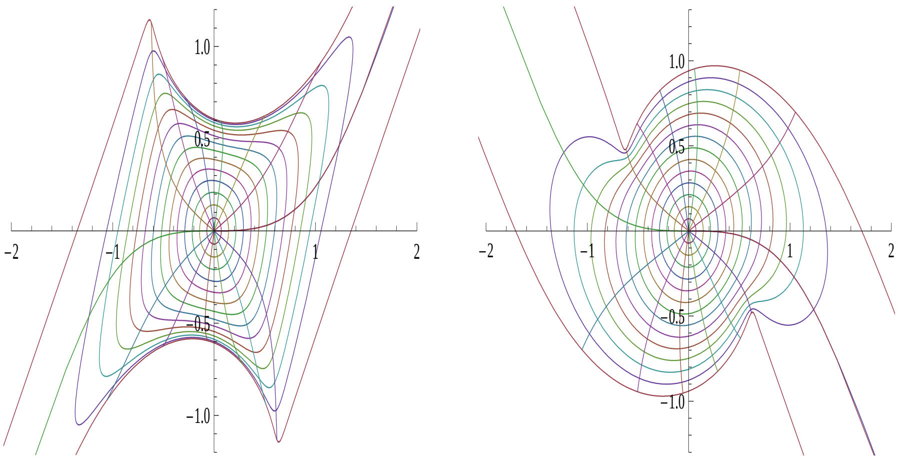
(b) $a = 1, b = 0.5, m = 4$ and $\alpha = -\frac{(1+i)}{\sqrt{2}}$



(c) $a = 1, b = 0.5, m = 6$ and $\alpha = 1$



(d) $a = 1, b = \frac{1}{7}, m = 4$ and $\alpha = \frac{i}{7}$



(e) $a = \frac{3}{4}, b = \frac{2}{3}, m = 2$ and $\alpha = -\frac{0.871(1+i)}{\sqrt{2}}$

FIGURE 3. Images of $f(z)$ and the corresponding conformal pre-shears for the indicated values of a, b, m and α

References

1. H. Bateman (Ed. by A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi), Higher Transcendental Functions, Vol. I, McGraw-Hill, New York, 1953.
2. J. G. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A.I. **9**(1984), 3–25.
3. M. Dorff, Convolutions of planar harmonic convex mappings, *Comp. Vari. Theo. Appl.* **45**(2001), 263–271.
4. P. Duren, Univalent Functions (Grundlehren der mathematischen Wissenschaften 259, New York, Berlin, Heidelberg, Tokyo), Springer-Verlag, 1983.
5. P. Duren, Harmonic Mappings in the Plane, Cambridge Tracts in Mathematics, **156**, Cambridge Univ. Press, Cambridge, 2004.
6. P. Greiner, Geometric properties of harmonic shears, *Comput. Methods Funct. Theory* **4**(1)(2004), 77–96.
7. W. Hengartner, and G. Schober, *On Schlicht Mappings to Domains Convex in One Direction*, Comment. Math. Helv. **45** (1970), 303-314.
8. D. Kalaj, S. Ponnusamy, and M. Vuorinen, *Radius of close-to-convexity of harmonic functions*, *Complex Var. Elliptic Equ.*, Revised.
9. H. Lewy, *On the nonvanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc. **42** (1936), 689–692.
10. P. T. Mocanu, *Sufficient conditions of univalency for complex functions in the class* C^1, Anal. Numér. Théor. Approx. **10**(1)(1981), 75–79.
11. S. Ponnusamy, T. Quach, and A. Rasila, *Harmonic shears of slit and polygonal mappings*, `http://arxiv.org/pdf/1201.2015.pdf`
12. S. Ponnusamy, and A. Sairam Kaliraj *On Harmonic Close-to-convex Functions*, Comput. Methods Funct. Theory, (2012), To appear.
13. S. Ponnusamy, H. Yamamoto and H. Yanagihara, *Variability regions for certain families of harmonic univalent mappings*, Complex Var. Elliptic Equ. (2011), in print.
14. M. S. Robertson, Analytic functions starlike in one direction, *Am. J. Math.* **58**(1936), 465–472.
15. W. C. Royster and M. Ziegler, Univalent functions convex in one direction, *Publ. Math. Debrecen* **23**(1976), 339–345.

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